THE P-ADAPTIVE BIEM VERSION IN ELASTOSTATICS

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Abstract—This paper introduces the p-adaptive version of the boundary element method as a natural extension of the homonymous finite element approach. After a brief introduction to adaptive techniques through their finite element formulation in elastostatics, the concepts are cast into the boundary element environment. Thus, the p-adaptive version of boundary integral methods is shown to be a generalization of already well known ideas. In order to show the power of these numerical procedures, the results of two practical analysis using both methods are presented.

INTRODUCTION

The p-adaptive version of the Finite Element Method (FEM) was first suggested by Zienkiewicz *et al.* [1] in 1970. Since then a great deal of scientific research has been done on it [2], and today we have already witnessed the implementation of this technique into commercially available computer codes [3,4]. Despite that, its advantages are not yet fully appreciated by the bulk of the Finite Element Community.

Basically, in a p-adaptive process, the analyst only defines a simple mesh ("base" mesh) and a tolerance for the solution's quality. The order of the approximation functions in this mesh is automatically and selectively increased according to the value of local error indicators that are computed inside an auto-adaptive loop. The process stops when a global error estimator falls below the tolerance (Figure 1).

The two main advantages are evident: (i) The method starts the analysis from a very coarse mesh, the one strictly necessary to define the geometry and the loading, thus reducing dramatically the effort during the input preparation stage; and (ii) The iterative nature of the solution process and the successive increasing in the interpolation functions' order give a consistent (in some sense "natural") way to the estimation of the computation's accuracy. This fact is extremely important since it allows the introduction of quality control in our calculations, and can make the results less "analyst-dependent."

On the other hand, the Boundary Element Method (BEM) has generated very much research during recent years [5], its most attractive feature being the reduction in the dimensionality of the domain to be analyzed (from 3D to 2D, or from 2D to 1D). In some of its aspects we can say that the BEM has benefited from the experience gained in the development of the FEM. Hence, once again, it seems reasonable to extend the ideas supporting the p-adaptive version of the FEM to the BEM. A p-adaptive formulation of the BEM would have, when compared with the conventional method, the same advantages that the p-adaptive version of the FEM has over its classical formulation.

In this paper we summarize an approach to the p-adaptive version of the BEM applied to elastostatics [6–8]. The basic ideas are the same as behind the p-adaptive finite elements, so we present firstly a brief introduction to the technique through its finite element version. Then, the concepts of p-adaptivity are cast into the boundary element environment, producing a p-adaptive version in a natural manner. Finally, in order to show the power of those numerical procedures, we analyze two practical problems using both p-adaptive techniques, the one based on the FEM and the one based on the BEM.

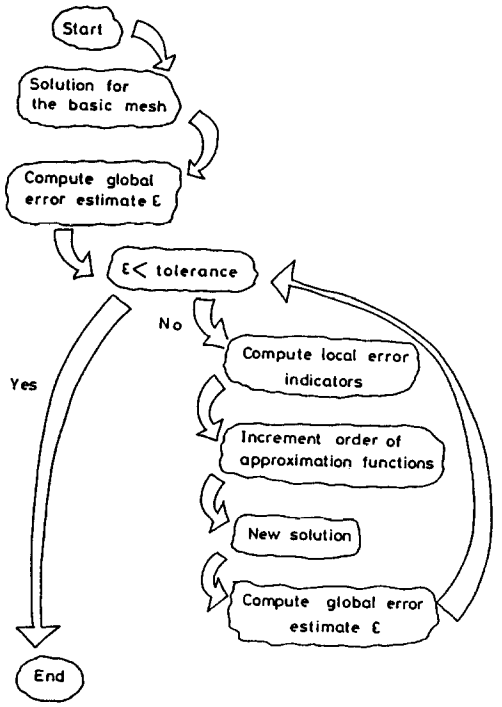

Figure 1. Flow chart of an auto-adaptive process.

P-ADAPTIVE FEM VERSION

The classical formulation of the FEM in linear elastostatics [9] takes the displacement field $\underset{\sim}{u}$ as the main unknown. Compatibility conditions in the domain of the problem Ω are automatically satisfied since the strain field $\underset{\sim}{\varepsilon}$ is defined starting from $\underset{\sim}{u}$. Equilibrium is imposed in a weak form through the virtual work principle, that can be written:

$$\int_{\Omega} \underset{\sim}{\omega} \cdot (\operatorname{div} \underset{\sim}{\sigma} + \underset{\sim}{f})\, d\Omega + \int_{\Gamma_t} \underset{\sim}{\omega} \cdot (\underset{\sim}{\bar{t}} - \underset{\sim}{\sigma} \cdot \underset{\sim}{n})\, d\Gamma = 0, \qquad \forall \underset{\sim}{\omega} \in \mathcal{P}, \tag{1}$$

where:

$\Gamma = \Gamma_u \bigcup \Gamma_t =$ boundary of Ω,

$\underset{\sim}{\sigma} =$ stress tensor ,

$\underset{\sim}{f} =$ prescribed body force field ,

$\underset{\sim}{n} =$ unit outward normal vector to Γ ,

$\underset{\sim}{\bar{t}} =$ prescribed boundary traction field on Γ ,

$\mathcal{P} =$ space of weighting functions (variations).

The compatibility requirements at boundary Γ_u are fulfilled a priori by taking an approximation for $\underset{\sim}{u}$ in the form:

$$\underset{\sim}{u} \cong \underset{\sim}{\hat{u}} = \sum_{j=1}^{n} a_j \underset{\sim}{\nu_j} + \underset{\sim}{\beta}, \tag{2}$$

where: $\underset{\sim}{\nu_j} \in \mathcal{P}$, $a_j \in \mathbf{R}$, $\underset{\sim}{\beta} = \underset{\sim}{\bar{u}}$ on Γ_u, and $\underset{\sim}{\bar{u}}$ are the prescribed boundary displacements on Γ_u.

The divergence theorem applied to equation (1) gives:

$$\int_\Omega \underset{\sim}{\sigma} : \text{ grad } \underset{\sim}{\omega}\, d\Omega = \int_\Omega \underset{\sim}{\omega} \cdot \underset{\sim}{f}\, d\Omega + \int_{\Gamma_t} \underset{\sim}{\omega} \cdot \underset{\sim}{\bar{t}}\, d\Gamma, \qquad \forall \underset{\sim}{\omega} \in \mathcal{P}. \tag{3}$$

Introducing in (3) the approximation (2) through the compatibility and constitutive relations, and taking as weighting functions $\underset{\sim}{\omega}$ the $\{\underset{\sim}{\nu}_j,\ j = 1, \ldots, n\}$, provides a linear set of n equations to compute the a_j coefficients:

$$\underset{\sim}{K}\, \underset{\sim}{a} = \underset{\sim}{b}. \tag{4}$$

In the classical version of the FEM the "trial" functions $\underset{\sim}{\nu}_j$ and $\underset{\sim}{\beta}$ are chosen so that the parameters a_j can be identified with the values of displacement components at certain points called nodes. Such an election gives physical meaning to these coefficients, but it has a basic drawback if a solution improvement is required. In this case, increasing the order of the approximation means necessarily, the redefinition of the mesh since at each node only the $\underset{\sim}{\nu}_j$ function should be non-zero. Hence, the stiffness matrix $\underset{\sim}{K}$ must be entirely recalculated, and previous computational effort cannot be effectively exploited.

In a p-adaptive approach, the first step is to establish a "hierarchy" among the $\underset{\sim}{\nu}_j$ approximating functions [10,11]. The idea is that the solution is improved by adding to the current set of functions those situated in the next level of hierarchy. Thus, since the current set of functions includes all the functions previously used, to compute the enhanced solution we only have to calculate a small quantity of new stiffness coefficients in $\underset{\sim}{K}$ and new "nodal" forces in $\underset{\sim}{b}$.

Furthermore, the hierarchy between approximations provides a "natural" way to construct local error indicators in order to steer the auto-adaptive process. These indicators play an important role, as they identify where a further refinement significantly improves the solution.

If the current approximation for the displacement field $\underset{\sim}{\hat{u}}$ is given by (2), and we are studying the usefulness of introducing a new function $\underset{\sim}{\nu}_{n+1}$, we may assume that the error $\underset{\sim}{e}$ in the subdomain Ω_{n+1}, where $\underset{\sim}{\nu}_{n+1}$ is non-zero, can be written as:

$$\underset{\sim}{e} = \underset{\sim}{u} - \underset{\sim}{\hat{u}} \cong a_{n+1}\, \underset{\sim}{\nu}_{n+1}, \tag{5}$$

provided that $\underset{\sim}{\nu}_{n+1}$ is in the next level of the established hierarchy [12]. Thus, the energy norm $\|\underset{\sim}{e}\|$ can be taken as an indicator of the error in Ω_{n+1}:

$$I = \|\underset{\sim}{e}\|^2 = \frac{1}{2} \int_{\Omega_{n+1}} \underset{\sim}{\sigma}^e : \underset{\sim}{\varepsilon}^e\, d\Omega, \tag{6}$$

where $\underset{\sim}{\sigma}^e$ and $\underset{\sim}{\varepsilon}^e$ are the stresses and strains generated by the error field $\underset{\sim}{e}$.

The indicators I with higher values point to the zones of Ω where it is more profitable to introduce the new functions.

Once the hierarchy between functions has been defined and the local error indicators have been constructed, the last key element of the adaptive process is the global error estimator. A great deal of scientific research has been done on the so called *a posteriori error estimates* [11,13,14] because their role is extremely important. They tell when the solution can be considered good enough, and they give a measure of the solution's quality. This kind of estimator should represent a reasonable approximation to the energy norm of the error field $\underset{\sim}{e}$. In practice three approaches are usually followed:

(a) The global error estimate is computed by adding the local error indicators (similar to those given by (6)) corresponding to a number of functions in the next hierarchical level [15].

(b) The estimator is based on a certain norm of the residuals [11]:

$$\begin{aligned} \underset{\sim}{r} &= \text{div}\ \hat{\underset{\sim}{\sigma}} + \underset{\sim}{f}, \qquad \text{in } \Omega, \\ \underset{\sim}{g} &= \bar{\underset{\sim}{t}} - \hat{\underset{\sim}{\sigma}} \cdot \underset{\sim}{n}, \qquad \text{on } \Gamma_t. \end{aligned} \tag{7}$$

(c) Some theoretical convergence relationships between the number of degrees of freedom and the energy error is exploited [16].

Whatever the method selected, the function of the estimator is clear: it controls the auto-adaptation process of the solution.

P-ADAPTIVE BEM VERSION

As is well known, the so-called "direct" BEM for elastostatics is based on Betti's formula:

$$\underset{\sim}{c} \cdot \underset{\sim}{u}(P) + \int_\Gamma \underset{\sim}{T}^*(P,Q) \cdot \underset{\sim}{u}(Q)\, d\Gamma(Q) = \int_\Gamma \underset{\sim}{U}^*(P,Q) \cdot \underset{\sim}{t}(Q)\, d\Gamma(Q), \tag{8}$$

expressing a reciprocity relationship between the actual tractions state $\underset{\sim}{t}(Q)$ and displacements state $\underset{\sim}{u}(Q)$ at point Q on the boundary, and a fundamental solution defined by a concentrated unit load acting at point P, whose mechanical state is reflected by tractions $\underset{\sim}{T}^*(P,Q)$ and displacements $\underset{\sim}{U}^*(P,Q)$. $\underset{\sim}{c}$ is a matrix related to local geometric properties of the boundary around the collocation point P.

Interpreting $\underset{\sim}{T}$ and $\underset{\sim}{U}$ as weighting functions, it is possible to interpolate $\underset{\sim}{u}$ and $\underset{\sim}{t}$ through the classical projective methods, in order to discretize the problem into a set of linear equations:

$$\underset{\sim}{K}\, \underset{\sim}{a} = \underset{\sim}{b}, \tag{9}$$

where $\underset{\sim}{a}$ represents the boundary unknowns and $\underset{\sim}{K}$ and $\underset{\sim}{b}$ are computed by numerical integrations of influence coefficients.

In the following paragraphs we present an approach to the p-Adaptive version of the procedure just described.

Hierarchial Interpolation Functions

As in the finite element case, the first step is the selection of a family of interpolation functions with a hierarchical relationship between them. After some numerical experimentation [8] we have decided to use the Legendre family, defined by:

$$\begin{aligned} P_0(\xi) &= \frac{1}{2}(1-\xi), \\ P_1(\xi) &= \frac{1}{2}(1+\xi), \end{aligned} \qquad (-1 \leqq \xi \leqq 1), \tag{10}$$

and

$$P_p(\xi) = \frac{1}{(p-1)!}\, \frac{1}{2^{p-2}}\, \frac{d^{p-2}}{d\xi^{p-2}} \left[(1-\xi^2)^{p-1}\right], \qquad \text{for } p \geqq 2,$$

with appropriate combinations based on tensorial products in the two dimensional boundary case.

We have found that this family produces a better conditioning of the influence coefficients matrix when functions of higher hierarchy are introduced.

Collocation Points

Collocation of (8) at a set of n points on the boundary yields the linear set of equations that gives the boundary unknowns.

In the particular case of the isoparametric BEM, the same set of points is used to define geometry and boundary conditions, as collocation points and to support interpolation functions. This fact produces undesirable effects when the method must fit into special local situations.

However, in the BEM p-adaptive version we have complete freedom to choose the collocation points. The criterion is to place these points where the interpolation functions have their maximum values, in order to reinforce the corresponding diagonal elements within the influence matrix. Also, collocation points should be as far apart as possible to avoid ill conditioning in the resulting system of equations [17]. Thus, in 3D problems, the bilinear functions are collocated at corner nodes, and odd/even higher order functions are collocated at nodes belonging to edges/elements respectively.

Geometry and Boundary Conditions

In the p-adaptive BEM version the geometry and boundary conditions are represented independently of the analysis process. For example, "serendipity" functions defined over double-curved surface elements can be used to describe both the shape and the loading.

Adaptive Process: Local Indicators and Global Estimators.

The p-adaptive process is absolutely similar to that described for the FEM. A first or "base" solution is produced, using linear or bilinear interpolation for all unknown variables. The idea is to introduce in successive steps more p-hierarchical functions, in such a way that, at each step, with a minimum number of new functions, a maximum accuracy is reached. As in the FEM version the process is controlled by local error indicators, that tell where it is more profitable to increment the interpolation order, and by global error estimators, which tell when the solution is accurate enough. Due to the hierarchy, in each refinement step the influence matrix $\underset{\sim}{K}$ includes the one corresponding to the previous step, and only a few new coefficients should be computed.

In the next paragraphs we shall extend the ideas presented for the FEM to establish local error indicators in this process. To fix ideas let us think of a Neumann type problem, i.e., a problem i which the displacement field $\underset{\sim}{u}$ is unknown around the whole boundary Γ and the boundary tractions $\underset{\sim}{\bar{t}}$ are known on Γ. If we use an approximation $\underset{\sim}{\hat{u}}$ for $\underset{\sim}{u}$, the collocation equations can be written:

$$\begin{aligned} \underset{\sim}{c} \cdot \underset{\sim}{\hat{u}} + \int_\Gamma \underset{\sim}{T}^* \cdot \underset{\sim}{\hat{u}} \, d\Gamma &= \underset{\sim}{b}, \\ \underset{\sim}{b} &= \int_\Gamma \underset{\sim}{U}^* \cdot \underset{\sim}{t} \, d\Gamma. \end{aligned} \tag{11}$$

These equations are satisfied only at collocation points, while at other points there will be a residual $\underset{\sim}{r}$:

$$\underset{\sim}{r} = \underset{\sim}{L} \, \underset{\sim}{\hat{u}} - \underset{\sim}{b}, \tag{12}$$

where the operator $\underset{\sim}{L}$ is such, that for a function $\underset{\sim}{F}(Q)$ defined along the boundary

$$\underset{\sim}{L} \, \underset{\sim}{F}(P) = \underset{\sim}{c} \cdot \underset{\sim}{F}(P) + \int_\Gamma \underset{\sim}{T}^*(P,Q) \cdot \underset{\sim}{F}(Q) \, d\Gamma. \tag{13}$$

In 2D elastostatics the operator $\underset{\sim}{L}$ can be written:

$$\underset{\sim}{L} \, \underset{\sim}{u} = \begin{bmatrix} L_1 \, \underset{\sim}{u} \\ L_2 \, \underset{\sim}{u} \end{bmatrix} = \begin{bmatrix} c_{11} \, u_1 + \int_\Gamma T_{11} \, u_1 \, d\Gamma + \int_\Gamma T_{12} \, u_2 \, d\Gamma \\ c_{22} \, u_2 + \int_\Gamma T_{21} \, u_1 \, d\Gamma + \int_\Gamma T_{22} \, u_2 \, d\Gamma \end{bmatrix}. \tag{14}$$

The relationship between the residual $\underset{\sim}{r}$ and the error $\underset{\sim}{e}$ is :

$$\underset{\sim}{L}\,\underset{\sim}{e} = \underset{\sim}{L}\,(\underset{\sim}{u} - \underset{\sim}{\hat{u}}) = -\underset{\sim}{r}. \tag{15}$$

That is, $\underset{\sim}{r}$ is related to $\underset{\sim}{e}$ in the same way as $-\underset{\sim}{b}$ to $\underset{\sim}{u}$.

A measure similar to the energy norm in the FEM can now be introduced for the global error:

$$\|\underset{\sim}{e}\|^2 = \int_\Gamma \underset{\sim}{e}\cdot\underset{\sim}{r}\,d\Gamma = -\int_\Gamma \underset{\sim}{e}\cdot\underset{\sim}{L}\,\underset{\sim}{e}\,d\Gamma. \tag{16}$$

As in the FEM, a local error indicator can be obtained assuming that the solution $\underset{\sim}{u}$ is almost equal to the current approximation $\underset{\sim}{\hat{u}}$ plus the next hierarchical contribution:

$$\underset{\sim}{u} = \underset{\sim}{\hat{u}} + a_{n+1}\,\underset{\sim}{\nu}_{n+1}. \tag{17}$$

The error is, then, approximated by:

$$\underset{\sim}{e} = a_{n+1}\,\underset{\sim}{\nu}_{n+1}, \tag{18}$$

and

$$\|\underset{\sim}{e}\|^2 = -a_{n+1}\int_\Gamma \underset{\sim}{\nu}_{n+1}\cdot\underset{\sim}{r}\,d\Gamma. \tag{19}$$

But,

$$\underset{\sim}{L}\,\underset{\sim}{e} = a_{n+1}\,\underset{\sim}{L}\,\underset{\sim}{\nu}_{n+1} = -\underset{\sim}{r}, \tag{20}$$

hence, using (19)

$$\|\underset{\sim}{e}\|^2 = a_{n+1}^2\int_\Gamma \underset{\sim}{\nu}_{n+1}\cdot\underset{\sim}{L}\,\underset{\sim}{\nu}_{n+1}\,d\Gamma, \tag{21}$$

and comparing (19) and (21):

$$a_{n+1} = -\left[\int_\Gamma \underset{\sim}{\nu}_{n+1}\cdot\underset{\sim}{r}\,d\Gamma\right] \Big/ \left[\int_\Gamma \underset{\sim}{\nu}_{n+1}\cdot\underset{\sim}{L}\,\underset{\sim}{\nu}_{n+1}\,d\Gamma\right], \tag{22}$$

so, the desired indicator is:

$$I = \|\underset{\sim}{e}\|^2 = \frac{\left[\int_\Gamma \underset{\sim}{\nu}_{n+1}\cdot\underset{\sim}{r}\,d\Gamma\right]^2}{\left[\int_\Gamma \underset{\sim}{\nu}_{n+1}\cdot\underset{\sim}{L}\,\underset{\sim}{\nu}_{n+1}\,d\Gamma\right]}. \tag{23}$$

If the collocation formula (11) is applied at the boundary point where the residual is being computed, it gives:

$$\underset{\sim}{c}\cdot\underset{\sim}{u}^{\text{comp}} + \int_\Gamma \underset{\sim}{T}^*\cdot\underset{\sim}{\hat{u}}\,d\Gamma = \underset{\sim}{b}, \tag{24}$$

where $\underset{\sim}{u}^{\text{comp}}$ will be different from the value of $\underset{\sim}{\hat{u}}$ at the point.

On the other hand, the residual $\underset{\sim}{r}$ is :

$$\underset{\sim}{r} = \underset{\sim}{c}\cdot\underset{\sim}{\hat{u}} + \int_\Gamma \underset{\sim}{T}^*\cdot\underset{\sim}{\hat{u}}\,d\Gamma - \underset{\sim}{b}. \tag{25}$$

Subtracting (24) from (25) yields:

$$\underset{\sim}{r} = \underset{\sim}{c} \cdot (\underset{\sim}{\hat{u}} - \underset{\sim}{u}^{\text{comp}}) \tag{26}$$

as an expression that shows how to compute the residual.

As regarding the other key element within the adaptive process, the global error estimator, we must say that very much research is still needed. Currently, the most popular measures are based on norms of the residual field $\underset{\sim}{r}$ [17]. We can quote, for example, the H^0 norm:

$$||E||_0 = \left[\int_\Gamma \underset{\sim}{r} \cdot \underset{\sim}{r}\, d\Gamma\right]^{1/2}. \tag{27}$$

If straight or plane elements are used, $||E||_0$ gives a direct measure of the error field $\underset{\sim}{e}$ for the Neumann-type problem just considered. In this particular case, the integral including $\underset{\sim}{T}^*(P,Q)$ is null in (13), when the collocation point is inside the element. Then, there exists a direct relation between $\underset{\sim}{r}$ and $\underset{\sim}{e}$:

$$-\underset{\sim}{r} = \underset{\sim}{\underset{\sim}{L}}\, \underset{\sim}{e} = \underset{\sim}{\underset{\sim}{c}}\, \underset{\sim}{e}. \tag{28}$$

Another estimator used in mixed problems is the summation of fluxes or the equilibrium of tractions over the whole boundary. The magnitude of the unbalanced flux or force after some normalization, can be used as a measure of the accuracy.

ILLUSTRATIVE EXAMPLES

In order to show the capabilities of p-adaptive methods and how they can work in practice, two typical 2D analyses are included here. Both cases are solved using the FEM and the BEM so that the results given can be used to compare their performance. The solutions considered "exact" have been obtained by using a very fine mesh of boundary elements.

Hollow Gravity Dam

This example is taken from Kelly *et al.* [11]. We consider the analysis of a gravity dam using the discussed adaptive algorithms. The base meshes used for both procedures are shown in Figures 2 and 3. Note that there is not a specially fine discretization around the singularities,

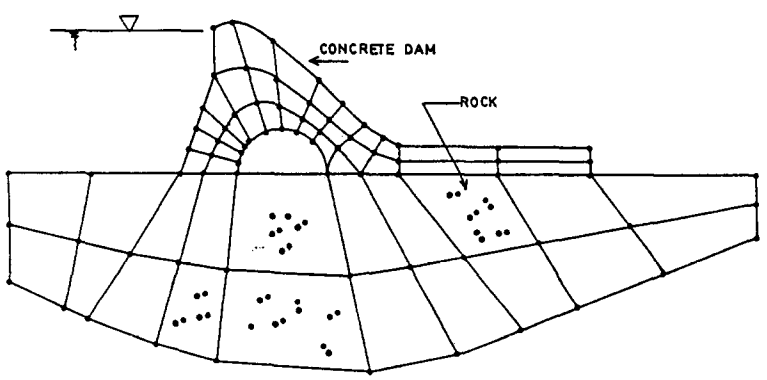

Figure 2. Hollow dam. Base mesh of finite elements.

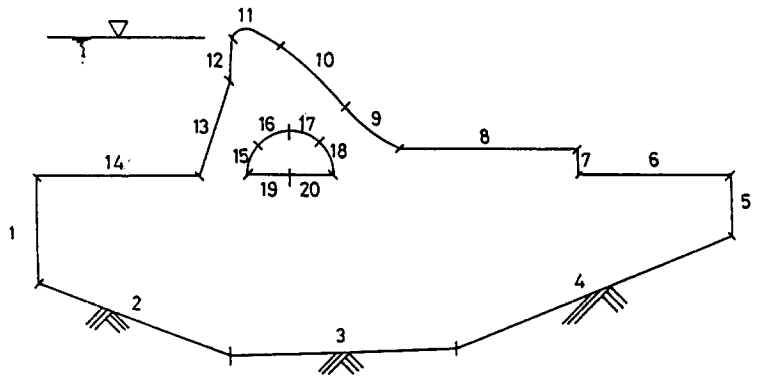

Figure 3. Hollow dam. Base mesh of boundary elements.

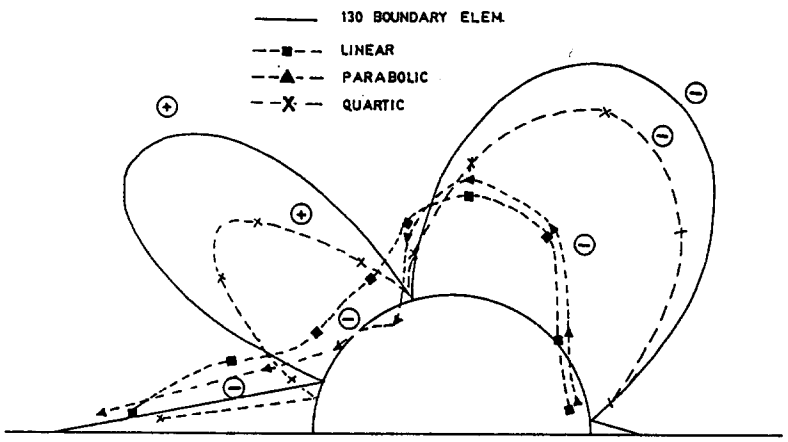

Figure 4. Hollow dam. Maximum principal stresses from auto-adaptive finite element analysis.

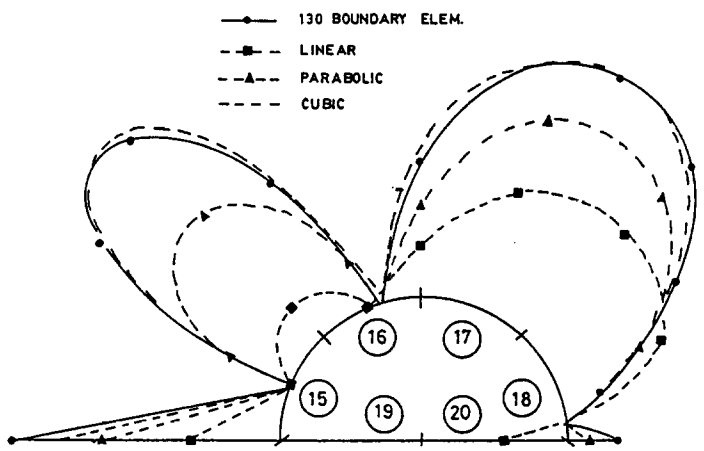

Figure 5. Hollow dam. Maximum principal stresses from auto-adaptive boundary element analysis.

as would be necessary in conventional analyses. The representative result chosen to show the performance of the methods is the maximum principle stress on the surface of the inner hollow gallery (Figures 4 and 5). It can be seen how, starting from a very poor approximation, in both cases the auto-adaptive process leads to the results considered "exact." The performance of the BEM is clearly better, probably due to the simultaneous and independent interpolation of both the tractions and the displacements. It is remarkable how the adaptive processes tend to correct the errors in the design of meshes.

Massive Gravity Dam.

As a second example we have chosen another dam, this time massive. The base meshes are shown in Figures 6 and 7 for the finite element and boundary element method, respectively. To compare the results we have selected the foundation pressures given by both procedures (Figures 8 and 9). The conclusions are basically the same as in the previous case.

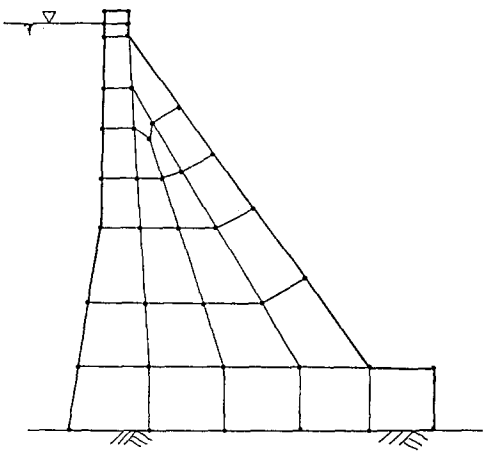

Figure 6. Massive dam. Base mesh of finite elements.

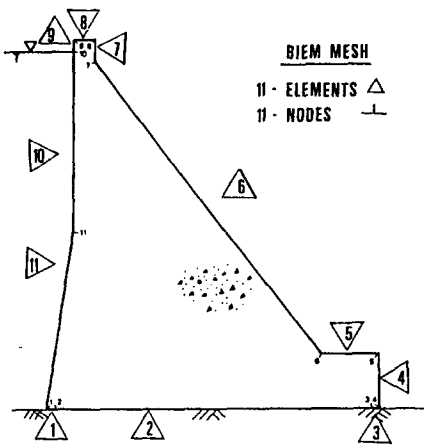

Figure 7. Massive dam. Base mesh of boundary elements.

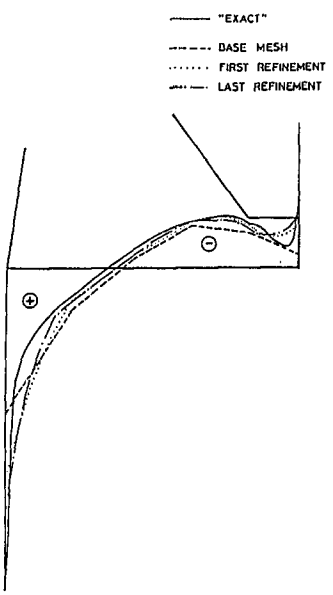

Figure 8. Massive dam. Foundation pressures from auto-adaptive finite element analysis.

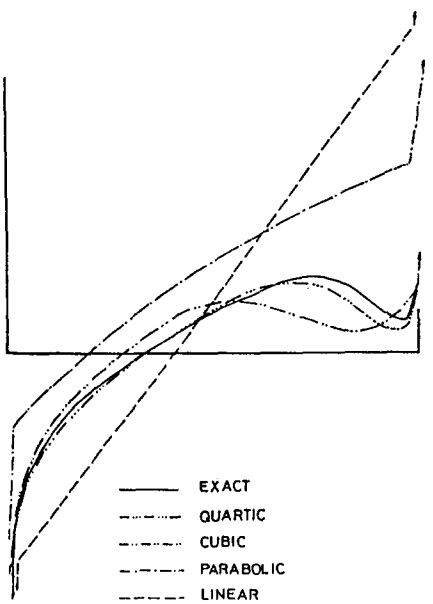

Figure 9. Massive dam. Foundation pressures from auto-adaptive boundary element analysis.

CONCLUSIONS

The p-adaptive version of the boundary element method has been introduced in this paper as an extension of the homonymous finite element approach. The advantages of these adaptive procedures are mainly two: the effort in data preparation and output processing is significantly reduced, and error estimators inherent in the adaptive algorithms allow for the introduction of quality measures in the results. The numerical examples presented here have illustrated the application of the technique to practical cases. It has been shown, for example, how the adaptive process minimizes the solution dependence on discretization errors. Thus, the same mesh can be used to analyze very different loadings.

Regarding the BEM version, although the local indicators and global estimators mentioned here have been used successfully, a great deal of research has been found necessary for solving a great number of critical aspects, especially related to the development of a reliable and computationally cheap global error estimator.

REFERENCES

1. O.C. Zienkiewicz, B.M. Irons, F.E. Scott and J.S. Campbell, High speed computing of elastic structures, *Proc. IUTAM. Liege* (1970).
2. I. Babuska, O.C. Zienkiewicz, J. Gago and E. Oliveira (eds.), *Accuracy Estimates and Adaptive Refinements in Finite Element Computations*, John Wiley, (1986).
3. A. Peano, General purpose systems based on adaptive finite elements. Software design considerations, *ISMES. Bergamo* (1984).
4. B. Taylor and S. Gupta, Control of the error of discretization and idealization in the new p-version finite element program PROBE, *Proc. 2nd. Int. Conf. on Supercomp. Appl. in Automative Industry. Sevilla* (1988).
5. C. Brebbia, J. Telles and L. Wrobel, *Boundary Element Techniques*, Springer-Verlag, Berlin, (1984).
6. E. Alarcon, A. Reverter and J. Molina, Hierarchical boundary elements, *Computers and Structures* **20**, 151–156 (1985).
7. E. Alarcon and A. Reverter, p-adaptive boundary elements, *Int. J. Num. Meth. Eng.* **23**, 801–829 (1986).
8. M. Cerrolaza and E. Alarcon, p-adaptive boundary elements for three-dimensional potential problems, *Com. Appl. Num. Meth.* **3**, 335–344 (1987).
9. T. Hughes, *The Finite Element Method*, Prentice Hall, (1987).
10. O.C. Zienkiewicz and K. Morgan, *Finite Elements and Approximation*, John Wiley, (1983).
11. D.Kelly, J. Gago, O.C. Zienkiewicz and I. Babuska, A posteriori error analysis and adaptive process in the FEM. Parts I and II, *Int. J. Num. Meth. Eng.* **19**, 1953–1619 (1983).
12. O.C. Zienkiewicz, J. Gago and D. Kelly, The hierarchical concept in finite element analysis, *Computers and Structures* **16** (1–4), 53–65 (1983).
13. I. Babuska and W. Rheinboldt, A posteriori error estimates for the finite element method, *Int. J. for Num. Meth. Eng.* **12**, 1597–1615 (1978).
14. I. Babuska, The p and h-p versions of the finite element method. The state of the art, *Technical Note BN-1156. University of Maryland* (1986).
15. P. Basu and A. Peano, Adaptivity in p-version finite element analysis, *ASCE Journal of Structural Eng.* **109** (10) (1983).
16. B. Szabo, Estimation and control of error based on p-convergence, *Accuracy Estimates and Adaptive Refinements in Finite Elements Computations (ARFEC), Lisbon* (1984).
17. M. Cerrolaza, p-adaptive boundary elements: development and applications in potential theory and elastostatics, Ph.D Thesis. *ETSI Industriales. Polytechnical Univ. of Madrid* (1988).